# Rigidity of spherical codes, and kissing numbers in high dimensions

Abhinav Kumar
joint work with Henry Cohn, Yang Jiao and Salvatore Torquato

MIT

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# Rigidity - Introduction

Recall (cf. Bob Connelly's talk) that a sphere packing is rigid if it admits no local deformations, i.e. we can't move the spheres continuously without making them overlap (apart from using isometries of the ambient space).

If we can show a packing is not rigid, it might lead the way toward improving the sphere packing for density.

We'll study rigidity for spherical codes, and describe results for some kissing configurations in low dimensions.

We'll use the terms *jammed* and *rigid* interchangeably.

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# Spherical codes

**Definition**

A spherical code $\mathcal{C}$ is a finite sets of points on the surface of a sphere $S^{n-1} \subset \mathbb{R}^n$.

A basic invariant is the minimal angular separation $\theta(\mathcal{C})$ between distinct points of $\mathcal{C}$.

Good spherical codes: have large angular distance between distinct points.

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# Spherical codes II

We may ask, given $N$, how to place the points of $C$ such that $\theta(C)$ is maximized.

Conversely, given $\theta_0$, what is the maximum number of points $N$ in a code $C$ with $\theta(C) \geq \theta_0$?

For $\theta_0 = \pi/3$, the latter problem becomes the **kissing number** problem.

Answers only known in dimensions $1, 2, 3$ (Schütte and van der Waerden), 4 (Musin), 8 and 24 (Odlyzko-Sloane and Levenshtein).

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# Sphere packings

**Definition**

A sphere packing in $\mathbb{R}^n$ is a collection of spheres/balls of equal size which do not overlap (except for touching). The density of a sphere packing is the volume fraction of space occupied by the balls.

A central question is to find a/the densest packing in $\mathbb{R}^n$.

# The usual suspects

In low dimensions, the best sphere packings seem to come from lattices, as do some of the best kissing configurations.

- The simplex lattice $A_n = \{x \in \mathbb{Z}^{n+1} : \sum x_i = 0\}$ in the zero-sum hyperplane in $\mathbb{R}^{n+1}$.
- The checkerboard lattice $D_n = \{x \in \mathbb{Z}^n : \sum x_i \text{ even } \}$.
- The special root lattice $E_8 = D_8 \bigcup \left(D_8 + (\frac{1}{2}, \dots, \frac{1}{2})\right)$.
- $E_7$, the orthogonal complement of an $A_1$ inside $E_8$.
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# Some records

The densest lattices in low dimensions are

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 24 |
|---|---|---|---|---|---|---|---|---|---|
| $\Lambda$ | $A_1$ | $A_2$ | $A_3$ | $D_4$ | $D_5$ | $E_6$ | $E_7$ | $E_8$ | Leech |

The best known kissing numbers in low dimensions are

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 24 |
|---|---|---|---|---|---|---|---|---|---|
| $\Lambda$ | 2 | 6 | 12 | 24 | 40 | 72 | 126 | 240 | 196560 |

But in most dimensions (e.g. 10) the best known packings or kissing numbers come from non-lattices.

# Rigidity

Say a spherical code is

- Rigid or jammed if the only continuous motions of its points on the sphere, such that the minimal distance does not drop below its initial value, consist of applying global isometries (i.e. are induced by a path in $SO(n)$ starting at the identity).
- Locally jammed if there is no continuous motion of one point of the code while keeping the others fixed, that does not decrease the minimal distance.
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# Linear programming

We can write a linear program to check whether any infinitesimal motions are allowed [Donev, Connelly, Stillinger, Torquato].

Let $x_1, \ldots, x_N$ be vectors describing the centers of $N$ spheres of radius $R$. If spheres $i$ and $j$ are adjacent, we have $|x_i - x_j| = R$.

We take an infinitesimal motion $x_i + ty_i$ of these sphere centers. The condition for this to be admissible is

$$|x_i - x_j + t(y_i - y_j)| \geq R$$

for $i, j$ in contact (the other constraints are irrelevant for $t$ small).

# Infinitesimal rigidity by LP

This simplifies to

$$\langle x_i - x_j, y_i - y_j \rangle \geq 0$$

to first order.

This is a linear condition in the coordinates of the $y_i$.

It can be shown that a packing in Euclidean space is infinitesimally rigid if and only if it is rigid (Roth, Whiteley, Connelly).

So we can test rigidity of sphere packings by an algorithm (among periodic packings with a fixed number of translates).

# LP for spherical codes

Essentially the same idea applies to spherical codes. Let $\mathcal{C}$ consist of $x_1, \dots, x_N$. We use deformation vectors $y_i$ which are in the tangent space. The infinitesimal constraint is that $\langle x_i, y_i \rangle = 0$, which is linear in the $y_i$.

The distance constraint is that $\langle x_i + ty_i, x_j + ty_j \rangle \leq \langle x_i, x_j \rangle$ whenever $x_i$ and $x_j$ are at the smallest distance. The first order condition is $\langle x_i, y_j \rangle + \langle x_j, y_i \rangle \leq 0$.

The linear program asks if some value $\langle x_i, x_j \rangle$ can be changed: the first order change in this quantity is $\langle x_i, y_j \rangle + \langle x_j, y_i \rangle$. This is the objective function.

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# Infinitesimal rigidity by LP

**Lemma**

*Suppose that $\delta(i,j) = \langle x_i, y_j \rangle + \langle x_j, y_i \rangle = 0$ for every $i \neq j$, for every infinitesimal deformation $\{y_i\}$ of the code $\mathcal{C} = \{x_i\}$. If the vectors $\{x_i\}$ span the ambient space $\mathbb{R}^n$, then $\mathcal{C}$ is infinitesimally jammed.*

We really have a collection of linear programs. If the objective function is always 0, we conclude that the spherical code is collectively jammed.

However, if we do get a nonzero answer, it does not necessarily mean that the code is not rigid. (Infinitesimal rigidity for spherical codes could be strictly stronger than rigidity)

Using dot products as above gets rid of orthogonal motions.

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Let's begin with the kissing configurations of the root lattices $A_n, D_n, E_6, E_7, E_8$.

**Example**

$A_2$: jammed, provably the best kissing configuration on $S^1$, using angles.

**Example**

$A_3$: locally jammed, but not collectively. In fact, one can move the points around to achieve any permutation while still maintaining the minimum distance.

Similarly, we can contruct an explicit unjamming of the $A_n$ kissing configuration for $n \geq 4$, using the one for $n = 3$.

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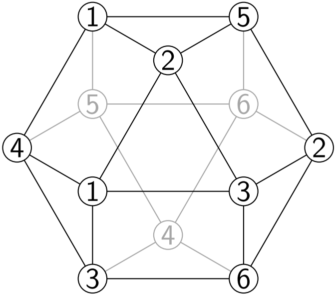
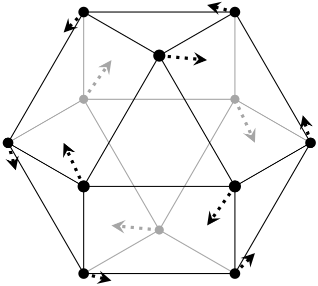
# $A_3$ unjamming by picture

Cuboctahedron:



Move pairs toward each other (twists the triangles).

# $D_4$

**Example**

The kissing configuration of $D_4$ is jammed.

**Proof.**

This spherical code has 24 points. Normalize them to have length $\sqrt{2}$. Distinct vectors have inner products in $\{0, \pm 1, -2\}$. If $\langle x, y \rangle = \pm 1$, then $x$ and $y$ span a copy of $A_2$, which is infinitesimally rigid. So their inner product does not change to first order. The $-2$ inner product occurs between antipodal vectors, so it does not change to first order.

If $\langle x, y \rangle = 0$, we can "connect" them by intermediate vectors which have inner product 1 with $x$ and $y$ and use these to show that $\langle x, y \rangle$ does not change. □

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# $D_n$, $n > 4$ and $E_6$, $E_7$, $E_8$

**Example**

The kissing configurations of $D_n$, $n > 4$ and $E_6$, $E_7$, $E_8$ are all jammed.

The proof is by "embedding" $A_2$ and $D_4$ into these lattices. The $\pm 1$ inner products do not change because the vectors concerned span a copy of $A_2$, and $A_2$ is jammed. Similarly, the 0 inner products do not change because for any two orthogonal vectors in $D_n$, there is a copy of $D_4$ inside $D_n$ containing them.

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# Competitors in $5, 6, 7$ dimensions

The best known kissing configurations in these dimensions are not unique.
There is one more 40-point kissing configuration in $\mathbb{R}^5$, competing with that of $D_5$.

There are three more 72-point kissing configuration in $\mathbb{R}^6$, competing with that of $E_6$.

There are three more 126-point kissing configuration in $\mathbb{R}^7$, competing with that of $E_7$.

Computer verification (rigorous, with rational numbers) shows they are all infinitesimally rigid.

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# $E_8$ and $\Lambda_9$

Next is the $E_8$ kissing configuration of 240 points. This is the unique best code of its minimal angle, so it is rigid.

**Example**

In 9 dimensions, the best kissing configuration coming from a lattice is that of the laminated lattice $\Lambda_9$. It consists of the 240 points from $E_8$ in the "equatorial" hyperplane, as well as points of the form $(0, \ldots, 0, \pm 1, 0, \ldots, 0, \pm 1)$.

These last 32 points lie above or below the "deep holes" of the $E_8$ kissing configuration, and they, along with the smallest vectors of $D_8 \subset E_8$, make up the root system of $D_9$.

# $\Lambda_9$

We know $D_9$ is infinitesimally jammed, so it's futile to try to move its minimal vectors. We are left with the half-integer vectors $(\pm\frac{1}{2}, \dots, \pm\frac{1}{2}, 0)$, with an even number of minus signs.

It turns out that these points are not even locally jammed, so we can move them out of the equatorial plane, showing that the kissing configuration of $\Lambda_9$ is not rigid.

In fact, can move half of these into the northern hemisphere simultaneously, and the other half into the southern hemisphere.

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# Other kissing configurations in $9, 10, 11, 12$ dimensions

$\mathbb{R}^9$: 306 points from the non-lattice packing $P_{9a}$.

$\mathbb{R}^{10}$: 500 points from the non-lattice packing $P_{10b}$.

$\mathbb{R}^{11}$: 582 points from the non-lattice packings $P_{11c}$.

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These are eleven such spherical codes in dimension 11 and seventeen in dimension 12. We proved that they are all jammed.

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# Coxeter-Todd

**Example**

The best known lattice kissing configuration in $\mathbb{R}^{12}$ is that of the Coxeter-Todd lattice $K_{12}$, which is the densest known lattice in that dimension. It has 756 minimal vectors.

The linear program gives a first-order unjamming of $K_{12}$, but the obvious motion $x_i \to (x_i + ty_i)/\sqrt{1 + t^2|y_i|^2}$ doesn't work. We can make a choice of first-order deformation, set up a linear program for a second order deformation, which also gives a non-trivial answer. But the obvious lift doesn't work ...

# Unjamming Coxeter-Todd

Thankfully, we can exploit the Eisenstein lattice structure of $K_{12}$ to unjam this kissing configuration. It splits into 126 disjoint hexagons, which are far enough apart that they can be rotated independently through small angles, without changing the minimal distance.

The key property is that for distinct minimal vectors $x, y$ of $K_{12}$ (considered as a $\mathbb{Z}[\omega]$ lattice inside $\mathbb{C}^6$, we have not just $\mathrm{Re}\,\langle x, y\rangle \leq 2$ but also the stronger property $\langle x, y\rangle = \sum x_i \overline{y_i} \leq 2$.

This fact is equivalent to an assertion about the 126-point code in $\mathbb{CP}^5$ obtained by taking the quotient by $\mathbb{C}^\times$ or by the sixth roots of unity.

We have shown that the kissing configuration of $K_{12}$ is not rigid.

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# Barnes-Wall lattice

**Example**

The densest lattice in $\mathbb{R}^{16}$ is the Barnes-Wall lattice, a laminated lattice. Its kissing arrangement of 4320 vectors is also the best known in this dimension.

The linear program was too large to run on a computer. However, we showed by using $A_2$ and $D_4$ embeddings, and the automorphism group of this lattice, that the kissing configuration is jammed, so no local improvements are possible.

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# Leech lattice

The Leech lattice kissing arrangement of 196560 vectors is an optimal code, unique for its size and minimal distance. Therefore it is certainly rigid.

The inner products between distinct vectors lies in $\{0, \pm 1/4, \pm 1/2, -1\}$.

# $\Lambda_{25}$

**Example**

The (previous) record for kissing number in $\mathbb{R}^{25}$ was $196560 + 96$, from the kissing configuration of the laminated lattice $\Lambda_{25}$. It consists of taking the Leech minimal vectors on the equatorial hyperplane, along with the remaining vectors of $D_{25}$ (a 24-dimensional cross-polytope above and below).

The kissing configuration of $\Lambda_{25}$ is unjammed, just like that of $\Lambda_9$. However, using this to try improve the kissing number is quite difficult.

# Dimension 25

We found two different ways to beat the kissing number in $\mathbb{R}^{25}$.

One way is to look for a large kissing configuration which still contains $\mathcal{C}_{\mathrm{Leech}}$ as a cross-section. This leads to a small improvement. We describe the other method, which easily generalizes to give improvements in higher dimensions.

Within $\mathcal{C}_{\mathrm{Leech}}$, we searched by computer (simulated annealing) for a subset $S$ such that for distinct $x$, $y$ in $S$, the inner product is never $1/2$, i.e. $\langle x, y \rangle \leq 1/4$.

The largest $S$ we found has size 480. In fact, smaller $S$ will also work. Here's one that's easy to describe.

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# 288 vectors

There is a copy of the Nordstrom-Robinson code of size 256 in the Barnes-Wall lattice (replace 0 by 1 and 1 by $-1$). In fact, together with the cross-polytope formed by twice the 32 standard coordinate directions and their negatives in $\mathbb{R}^{16}$, these generate the Barnes-Wall lattice.

Since the Barnes-Wall lattice is a cross-section of the Leech lattice, we obtain an $S$ of cardinality 288. Then let

$$\mathcal{C}' = \Big\{(x,0) : x \in \mathcal{C}_{\mathrm{Leech}} \backslash S\Big\} \bigcup \Big\{(x\cos\theta, \pm 2\sin\theta) : x \in S\Big\}.$$

For $1/4 \leq \sin^2\theta \leq 1/3$, this code has minimal angle at least $\pi/3$.

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# Dimensions 26 through 31

We beat the previous kissing numbers in all these dimensions (which are all from laminated lattices starting from Leech).

**Example**

In dimension 26, we obtain a kissing number of $196560 + 4 \cdot 480$, which beats the previous record by 768.

This uses the following fact:

**Lemma**

*There exist two disjoint $1/2$-avoiding sets $S_1$ and $S_2$ of size 480 in $\mathcal{C}_{Leech}$.*

# Independent sets

Then let

$$C'' = \left\{(x,0) : z \in \mathcal{C}_{\mathrm{Leech}} \backslash (S_1 \cup S_2)\right\}$$
$$\bigcup \left\{(x\cos\theta, 2\omega^j \sin\theta) : x \in S_1, j = 0,2,4\right\}$$
$$\bigcup \left\{(x\cos\theta, 2\omega^j \sin\theta) : x \in S_2, j = 1,3,5\right\}$$

Here $\omega$ is a primitive sixth root of unity in $\mathbb{C}$ considered as $\mathbb{R}^2$, and $\sin\theta = 1/\sqrt{3}$.

**Proof.**

Probabilistic method! Let $S_1$ be fixed. Then the expected number of elements of $S_1 \bigcap gS_1$ (where $g$ runs over the elements of the automorphism group of the Leech lattice) is $480^2/196560 \approx 1.17$. So there exists $S_2 = gS_1$ which intersects $S_1$ in at most one element. $S_1$ can be chosen antipodal, so in fact they don't intersect at all. $\square$

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# Some open questions

1. Find an example of a spherical code which is rigid but not infinitesimally rigid.
2. Find an algorithm to test whether a spherical code is rigid.
3. Are there optimal (or near-optimal) kissing configurations in high dimensions which have no contacts at all?
4. Are there exponentially large (in the dimension) kissing configurations which are jammed?

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Reference: "Rigidity of spherical codes", Henry Cohn, Yang Jiao, Abhinav Kumar and Salvatore Torquato, arXiv:1102.5060.

# Thank you!