Equations for Hilbert modular surfaces

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Introduction

Outline of talk

- Elliptic curves, moduli spaces, abelian varieties

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- Applications
- Method/proof

Elliptic curves

An elliptic curve over $\mathbb{C}$ is the set of solutions to an equation

$$y^2 = x^3 + Ax + B$$

with $A, B \in \mathbb{C}$ with $\Delta = -4A^3 - 27B^2 \neq 0$.

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Geometrically, it's a complex torus $\mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is a lattice in $\mathbb{C}$.

The periods ω_1 and ω_2 can be computed as elliptic integrals.

Uniformization and moduli space

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Parametrized by $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$: write $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$.

Two values of τ give equivalent lattices iff related by action of $\mathrm{PSL}_2(\mathbb{Z})$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}$$

Modular function

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$$j : \mathsf{PSL}_2(Z)\backslash\mathbb{H} \xrightarrow{\sim} \mathbb{P}^1(\mathbb{C})$$

where j is the modular function

$$j(z) = \frac{1}{q} + 744 + 196884q + \dots$$

with $q = e^{2\pi i z}$.

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In general, elliptic curves and their moduli spaces are important in arithmetic (we can do a lot of this over $\mathbb{Q}$).

Similarly, pairs of elliptic curves with a map (isogeny) of degree N (with cyclic kernel) are also parametrized by a nice moduli space $\Gamma_0(N)\backslash\mathbb{H}$, where

$$\Gamma_0(N) = \{g \in SL_2(\mathbb{Z}) : g \equiv \mathrm{Id} \pmod{N}\}.$$

Classical modular polynomials

The curve $X_0(N) = \Gamma_0(N)\backslash\mathbb{H}$, can be given as a plane curve by the modular polynomial $\Phi_n(X, Y)$. These can quickly get very complicated. For instance, for $N = 2$ we have

$$\begin{aligned}\Phi_2(X, Y) = X^3 - X^2Y^2 + 1488X^2Y - 162000X^2 + 1488XY^2 \\ + 40773375XY + 8748000000X + Y^3 - 162000Y^2 \\ + 8748000000Y - 157464000000000\end{aligned}$$

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Here X, Y are the j-invariants of the two elliptic curves involved. Not so easy to guess that this is a genus 0 curve!

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where $j_2(z) = (\eta(q)/\eta(q^2))^{24}$, with

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$$X = \frac{(j_2 + 256)^3}{j_2^2}, \qquad Y = \frac{(j_2 + 16)^3}{j_2}.$$

Abelian varieties, moduli spaces

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The moduli space of abelian varieties is the quotient of Siegel upper half space $\mathbb{H}_g$, modulo some subgroup of the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$, depending on the polarization, level structure, etc.

For example, the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g is $\mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$.

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Another piece of data that can be used to cut down the size of the moduli space is the endomorphism ring.

Moduli spaces and arithmetic

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and so on.

Real multiplication

Some of the simplest examples beyond dimension one are given by abelian surfaces with real multiplication, i.e. a 2-dimensional abelian variety A such that its endomorphism ring is an order of a real quadratic field $\mathbb{Q}(\sqrt{D})$.

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Restricting to the simplest possible situation, we want to consider ppas (principally polarized abelian surfaces) A with endomorphisms by the full ring of integers $\mathcal{O}_D$ of $\mathbb{Q}(\sqrt{D})$.

Humbert surfaces

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As a complex variety, $Y_{-}(D)$ is $\mathrm{PSL}_2(\mathcal{O}_D)\backslash\mathbb{H}^+ \times \mathbb{H}^-$, where $\mathbb{H}^+$ is the upper half plane, and $\mathbb{H}^-$ is the lower half plane.

Geometric properties

The action of $g \in \mathrm{SL}_2(\mathcal{O}_D)$ is by considering the two embeddings $\mathbb{Q}(\sqrt{D}) \hookrightarrow \mathbb{R}$, and considering the action of $\mathrm{SL}_2(\mathbb{R})$ acting on $\mathbb{H}^+$ or $\mathbb{H}^-$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z_1, z_2) = \left(\frac{\sigma_1(a) z_1 + \sigma_1(b)}{\sigma_1(c) z_1 + \sigma_1(d)}, \frac{\sigma_2(a) z_2 + \sigma_2(b)}{\sigma_2(c) z_2 + \sigma_2(d)} \right)$$

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Hirzebruch, Zagier and van de Ven, in the 1970s, determined many geometrical properties of these surfaces. However, equations for very few of these were known (only for $D = 2$ or 5).

Equations for Hilbert modular surfaces

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But arithmetic models (over $\mathbb{Q}$) were not known.

Main result (Elkies - Kumar): We give a method to compute birational models for Hilbert modular surfaces over $\mathbb{Q}$, along with the map to $\mathcal{A}_2$. We computed equations for the first thirty non-trivial fundamental discriminants (i.e. $1 < D < 100$).

Example: $D = 17$

The equation of the Hilbert modular surface $Y_{-}(17)$, as a double cover of $\mathbb{P}^2$, is

$$z^2 = -256h^3 + (192g^2 + 464g + 185)h^2 - 2(2g+1)(12g^3 - 65g^2 - 54g - 9)h + (g+1)^4(2g+1)^2.$$

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The equation $h = -(6g^2 + g - 1)/8$ gives a non-modular curve on the Hilbert modular surface, and there is a family of genus 2 curves defined over it.

Some applications

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- Teichmüller curves on Hilbert modular surfaces.

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But there are examples of elliptic curves over quadratic fields with good reduction everywhere.

Tate's example

For instance, the following example due to Tate (analyzed by Serre). Let $K = \mathbb{Q}(\sqrt{29})$ and $\epsilon = (5 + \sqrt{29})/2$ a fundamental unit.

The elliptic curve

$$y^2 + xy + \epsilon^2 y = x^3$$

has discriminant $-\epsilon^{10}$, and so has good reduction everywhere.

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Basic idea: search the Hilbert modular surface $Y_{-}(D)$ for $Q(\sqrt{E})$-rational points.

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Basic idea: search the Hilbert modular surface $Y_{-}(D)$ for $Q(\sqrt{E})$-rational points.

But we need to know where to look. The search is guided by Hilbert modular forms and the Eichler-Shimura conjecture.

Eichler-Shimura

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But not known, for instance, if the field is real quadratic and the level is 1 !

Matching Frobenius

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We find such newforms in many instances, and these indicate where and how to search (match the Frobenius data).

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For instance, there is a newform of level 1 over the real quadratic field $\mathbb{Q}(\sqrt{193})$, with coefficients in $\mathbb{Q}(\sqrt{17})$.

The conjecture says this should correspond to an abelian surface over $\mathbb{Q}(\sqrt{193})$ with real multiplication by an order in $\mathbb{Q}(\sqrt{17})$ and good reduction everywhere.

Example

And here it is: let $w = (1+\sqrt{193})/2$, and let C be given by

$$y^2 + Q(x)y = P(x)$$

with

$$\begin{aligned} P(x) &= 2x^6 + (-2w+7)x^5 + (-5w+47)x^4 + (-12w+85)x^3 \\ &\quad + (-13w+97)x^2 + (-8w+56)x - 2w + 1, \\ Q(x) &= -x - w. \end{aligned}$$

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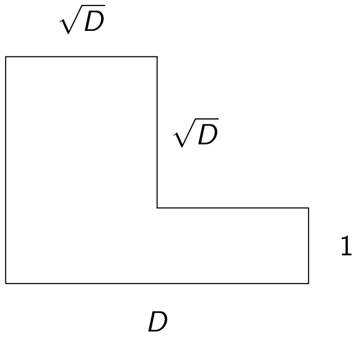
We produce evidence for the Eichler-Shimura conjecture, and a conjecture of Brumer and Kramer on paramodular forms.

Teichmüller curves

Consider billiards on the polygon below



Every billiard path on L_D is either closed or uniformly distributed with respect to Lebesgue measure.

Weierstrass curve

This is related to a particular property of an algebraic curve W_D: the moduli space of $(X, [\omega])$ where X is a Riemann surface whose Jacobian has real multiplication by $\mathcal{O}_D$, and ω is a holomorphic one-form with double zero on X. Then the immersion

$$W_D \to \mathcal{M}_2$$

is algebraic and isometric for the Kobayashi metrics on the domain and range. (work of McMullen)

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Many geometrical properties of W_D are known by work of McMullen, Bainbridge etc.

Equations for Teichmuller curves

In joint work with Ronen Mukamel, we produce equations cutting out W_D on the Hilbert modular surface $Y_-(D)$. Previously, some small discriminants were known by ad-hoc methods (work of Bouw, Moller and Zagier).

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For instance, the smallest D for which W_D has genus 1 is $D = 44$.

Theorem

The Weierstrass curve W_{44} is birational to the curve

$$y^2 = x^3 + x^2 + 160x + 3188.$$

(conductor 880, MW group over $\mathbb{Q}$ is cyclic, generated by $(26, 160)$).

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To actually prove that we have the right answer, one can produce bounds on the degree (using an appropriate modular form) of the curve, and exhibit sufficiently many exact points on the curve (by using correspondences to exhibit the extra endomorphisms: real multiplication).

Actual equation for W_{44}

In the [EK] coordinates on $Y_{-}(44)$, the Weierstrass curve is cut out by the equation $f_W(r,s)=0$, where

$$\begin{aligned} f_W(r,s) = {} & 196r^5 - 604r^6 + 637r^7 - 282r^8 + 45r^9 + 9604r^4 s \\ & - 16928r^5 s + 11379r^6 s - 3510r^7 s + 426r^8 s - 35476r^3 s^2 \\ & + 37908r^4 s^2 - 17004r^5 s^2 + 3752r^6 s^2 - 140r^7 s^2 \\ & + 46844r^2 s^3 - 22616r^3 s^3 + 348r^4 s^3 + 40r^5 s^3 \\ & + 1000r^6 s^3 - 26656rs^4 + 2240r^2 s^4 + 4640r^3 s^4 + 5488s^5. \end{aligned}$$

Method to produce equations for HMS

Step 1:

First, we attempt to compute the Humbert surface $\mathcal{H}_D$. It turns out to be birational to a moduli space of elliptic K3 surfaces, whose Neron-Severi lattice has signature $(1, 17)$ and discriminant $-D$.

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First, we attempt to compute the Humbert surface $\mathcal{H}_D$. It turns out to be birational to a moduli space of elliptic K3 surfaces, whose Neron-Severi lattice has signature $(1, 17)$ and discriminant $-D$.

Point: the elliptic K3 surfaces are easier to parametrize, because we can reverse engineer Tate's algorithm.

For all the discriminants $1 < D < 30$, the Humbert surface is birational to $\mathbb{P}^2$ (which makes life a little easier).

Say r, s are the two parameters on the surface $\mathcal{H}_D \sim \mathbb{P}^2$.

Method contd.

Step 2:

On each of the K3 surfaces (as a family over $\mathbb{P}^2_{r,s}$), find an alternate elliptic fibration which has reducible fibers of type E_8 and E_7 at $t = \infty$ and 0 respectively.

That is, the new Weierstrass equation looks like

$$y^2 = x^3 + t^3(at + a')x + t^5(b''t^2 + bt + b').$$

Method contd.

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On each of the K3 surfaces (as a family over $\mathbb{P}^2_{r,s}$), find an alternate elliptic fibration which has reducible fibers of type E_8 and E_7 at $t = \infty$ and 0 respectively.

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Step 3:

Use [K, 2006] to write the map to $\mathcal{M}_2$ or $\mathcal{A}_2$ in terms of the coeffients a, a', b, b', b''.

Method contd.

Step 4:

Figure out the branch locus of the 2:1 map $Y_{-}(D) \to \mathcal{H}_D$ geometrically by computing where the rank of the K3 surface jumps from 18 to 19, while its discriminant becomes $2D$ or $D/2$ (it acquires quaternionic multiplication).

Method contd.

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Figure out the branch locus of the 2:1 map $Y_{-}(D) \to \mathcal{H}_D$ geometrically by computing where the rank of the K3 surface jumps from 18 to 19, while its discriminant becomes $2D$ or $D/2$ (it acquires quaternionic multiplication).

Step 5:

Figure out the correct arithmetic twist by counting points on genus 2 curves, and using the Grothendieck-Lefschetz trace formula for Frobenius to figure out its characteristic polynomial.

Method contd.

Step 4:

Figure out the branch locus of the 2:1 map $Y_{-}(D) \to \mathcal{H}_D$ geometrically by computing where the rank of the K3 surface jumps from 18 to 19, while its discriminant becomes $2D$ or $D/2$ (it acquires quaternionic multiplication).

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Figure out the correct arithmetic twist by counting points on genus 2 curves, and using the Grothendieck-Lefschetz trace formula for Frobenius to figure out its characteristic polynomial.

Step 6: Write the equation of the Hilbert modular surface in the form

$$z^2 = f(r, s).$$

Rejoice.

Thank you!