

# Rational elliptic surfaces with high Mordell-Weil rank and multiplicative reduction

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# Elliptic surfaces

Let  $k$  be a subfield of  $\mathbb{C}$  (e.g.  $\mathbb{Q}$ ) and let  $E$  be an elliptic curve over  $K = k(t)$ . That is, genus 1 curve *with a point*.

Would like to understand the  $K$ -rational points of  $E$ , i.e. Mordell-Weil group  $E(K)$ .

Let  $\mathcal{E}$  be a relatively minimal proper model of  $E$ . We have an elliptic fibration (with section)  $\pi : \mathcal{E} \rightarrow \mathbb{P}_t^1$  whose generic fiber is  $E$ , and  $E(K)$  is in bijection with the sections of  $\pi$ .

The surface  $\mathcal{E}$  is an **elliptic surface**. The parameter  $n = \chi(\mathcal{E}, \mathcal{O}_{\mathcal{E}})$  measures the complexity of the surface.

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## Elliptic surfaces II

A Weierstrass equation (minimal for degrees) for  $\mathcal{E}$  looks like

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)$$

where  $a_i \in k[t]$  is a polynomial of degree at most  $n \cdot i$ .

- $n = 0$ : the elliptic fibration is trivial (i.e.  $\mathcal{E} = E_0 \times \mathbb{P}^1$ ). We'll usually assume this is not the case.
- $n = 1$ :  $\mathcal{E}$  is a rational surface (i.e. birational to  $\mathbb{P}^2$ , although over  $\bar{k}$ ).
- $n = 2$ :  $\mathcal{E}$  is a K3 surface.
- $n > 2$ :  $\mathcal{E}$  is honestly elliptic (i.e. Kodaira dimension is 1).

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# Rational elliptic surfaces

In this talk we'll only be concerned with **rational elliptic surfaces**. Even in this “simple” case there are lots of interesting phenomena.

The basic facts: for each reducible fiber, the non-identity components give rise to (the negative of a) root lattice  $A_n, D_n, E_r$  in the Néron-Severi or Picard group, which may be explicitly computed using Tate's algorithm.

We also have the class of the identity section and the fiber. These together with the root lattices above generate the trivial lattice  $T$ .

Theorem (Shioda-Tate)

*The Mordell-Weil group  $E(K) \cong NS(\mathcal{E})/T$ .*

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## Rational elliptic surfaces II

Oguiso and Shioda (1991) classified the possible Mordell-Weil lattices and reducible fiber configurations that could arise from rational elliptic surfaces.

This is possible since  $NS(\mathcal{E}) \cong H^2(\mathcal{E}, \mathbb{Z}) \cong U \oplus E_8(-1)$  in the rational case.

(For the analogous computation for elliptic K3 surfaces, see Shimada's massive 114-page paper, the last 92 of which are tables ...)

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## An $A_2$ example

Consider the elliptic surface

$$y^2 = (x - b_1)(x - b_2)(x - b_3) + t^2$$

where  $b_1 + b_2 + b_3 = 0$  (we may consider  $b_i \in \mathbb{Q}$ , or  $b_1, b_2$  independent indeterminates).

It's a rational elliptic surface, so Picard number is 10. Also, it has an  $E_6$  fiber at  $t = \infty$ . Therefore, generically (in  $b$ ) the Mordell-Weil rank is 2, with the Mordell-Weil lattice being isomorphic to  $A_2^*$ . The sections

$$P_i = (b_i, t)$$

of height  $2/3$  span the MWL, with  $P_1 + P_2 + P_3 = 0$ . Note that the symmetry group  $S_3$  of  $A_2^*$  acts on the parameters  $b_i$ , and the ring of invariants is exactly generated by the coefficients in the Weierstrass equation.

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## $E_8$ with additive reduction

### Example

If  $\mathcal{E}$  has a fiber of type II (i.e. a cusp), we may write its equation as

$$y^2 = x^3 + x\left(\sum_{i=0}^3 p_i t^i\right) + \sum_{i=0}^3 q_i t^i + t^5$$

We can transform to this as follows: put II fiber at  $t = \infty$ , then shift  $t$  to make  $q_4 = 0$ , and then scale  $t$  to make the coefficient of  $t^5$  equal to 1.

Note that there is still a scaling left (for  $x, y$ ) so this is really only a 7-parameter family. It helps a lot in the subsequent analysis.

Assume the coefficients  $(p_0, \dots, q_3)$  are *generic*.



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## $E_8$ with additive reduction: MW group

The Mordell-Weil group is 8-dimensional; with the Néron-Tate or canonical height pairing it is isometric to  $E_8$ .

There are precisely 240 sections of height 2; these are given by

$$(x, y) = (gt^2 + at + b, ht^3 + ct^2 + dt + e),$$

where these coefficients  $a, b, c, d, e, g, h$  satisfy:

$$h^2 = g^3$$

$$2ch = 1 + 3ag^2 + p_3g$$

$$c^2 + 2dh = 3a^2g + 3bg^2 + p_2g + p_3a$$

$$2cd + 2eh = a^3 + 6abg + p_1g + p_2a + p_3b + q_3$$

$$d^2 + 2ce = 3a^2b + 3b^2g + p_0g + p_1a + p_2b + q_2$$

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# Elementary symmetric polynomials and Weierstrass coefficients

To solve this system, we let  $u = g/h$ , then  $u$  is the specialization of the section at the additive fiber at  $t = \infty$ , which is  $\cong \mathbb{G}_a$ .

We can eliminate everything in terms of  $u$ , which satisfies a monic 240-degree equation with coefficients in  $k[p_0, p_1, \dots, q_3]$ . It turns out that we can attach weights to  $x, y, t, p_0, \dots, q_3, a, \dots, h, u$  so the system of equations is homogeneous.

Let  $u_1, \dots, u_{240}$  be the roots of this polynomial, and  $\sigma_j$  be the  $j$ 'th elementary polynomial.

## Theorem (Shioda)

*There is an explicit set of (invertible, non-linear) equations relating  $(p_0, p_1, \dots, q_3)$  and  $(\sigma_2, \sigma_8, \sigma_{12}, \sigma_{14}, \sigma_{18}, \sigma_{20}, \sigma_{24}, \sigma_{30})$ .*

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## ... and the invariants of the Weyl group

These numbers are the degrees of the fundamental invariants of the Weyl group of  $E_8$ .

In fact,  $G = W(E_8)$  is generically the Galois group of the splitting field extension  $k(u_1, u_2, \dots, u_8)/k(p_0, \dots, q_3)$  (choosing  $u_1, \dots, u_8$  to be the specializations at  $\infty$  of a basis of the Mordell-Weil group).

More is true: even on the level of the polynomial ring,  
 $k[u_1, \dots, u_8]^G = k[p_0, \dots, q_3]$ .

This family of rational elliptic surfaces is called an **excellent family**.

Key idea: the specialization map at infinity is a group homomorphism  
 $MW(X/\mathbb{P}^1) \rightarrow \mathbb{G}_a$ .

That is,  $u(P + Q) = u(P) + u(Q)$ .

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## Excellent family of elliptic curves

Let  $\mathcal{X} \rightarrow \mathbb{A}^n$  be a family of algebraic varieties, varying with respect to  $n$  parameters  $\lambda_1, \dots, \lambda_n$ . The generic member of this family  $X_\lambda$  is a variety over the rational function field  $k_0 = \mathbb{Q}(\lambda)$ .

Also, suppose that  $\mathcal{C}(X_\lambda)$  is a group of algebraic cycles on  $X_\lambda \times_{k_0} k$ , where  $k = \overline{k_0}$  is the algebraic closure, stable under the Galois group  $\text{Gal}(k/k_0)$ . Suppose also that there is an isomorphism  $\phi_\lambda : \mathcal{C}(X_\lambda) \otimes \mathbb{Q} \cong V$  for a fixed vector space  $V$ .

Then we have the Galois representation

$$\rho_\lambda : \text{Gal}(k/k_0) \rightarrow \text{Aut}(\mathcal{C}(X_\lambda)) \rightarrow \text{Aut}(V).$$

We let  $k_\lambda$  be the fixed field of the kernel of  $\rho_\lambda$ , i.e. it is the smallest extension of  $k_0$  over which the cycles of  $\mathcal{C}(X_\lambda)$  are defined. We call it the **splitting field** of  $\mathcal{C}(X_\lambda)$ .

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## Excellent family II

Now let  $G$  be a finite reflection group acting on the space  $V$ .

### Definition

We say  $\{X_\lambda\}$  is an **excellent family with Galois group  $G$**  if the following conditions hold:

- 1 The image of  $\rho_\lambda$  is equal to  $G$ .
- 2 There is a  $\text{Gal}(k/k_0)$ -equivariant evaluation map  $s : \mathcal{C}(X_\lambda) \rightarrow k$ .
- 3 There exists a basis  $\{Z_1, \dots, Z_n\}$  of  $\mathcal{C}(X_\lambda)$  such that if we set  $u_i = s(Z_i)$ , then  $u_1, \dots, u_n$  are algebraically independent over  $\mathbb{Q}$ .
- 4  $\mathbb{Q}[u_1, \dots, u_n]^G = \mathbb{Q}[\lambda_1, \dots, \lambda_n]$ .

## Excellent family III

We have just produced an excellent family of type  $E_8$ , i.e. for the Weyl group  $E_8$ . The group of algebraic cycles is just the Mordell-Weil lattice. The Galois equivariant evaluation map is the specialization homomorphism.

We may now use these formulas to produce number fields with Galois group  $W(E_8)$ , elliptic surfaces with rank 8, etc. (e.g. work of Shioda, Varilly-Alvarado – Zywina).



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# Multiplicative reduction

If we write down a random elliptic surface

$$y^2 = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t),$$

(i.e. choose the coefficients of the  $a_i$  randomly in  $[-N, N]$  for some large  $N$ ), then with probability close to 1 it will have no reducible fibers, and also that it will not have additive reduction anywhere.

Therefore, the multiplicative reduction case is much more fundamental.

What can we say in the multiplicative reduction case?

Since we're going to have 12 singular multiplicative fibers generically, we assume one is at  $\infty$ .

After some simple Weierstrass transformations, assume the equation is

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If we write down a random elliptic surface

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(i.e. choose the coefficients of the  $a_i$  randomly in  $[-N, N]$  for some large  $N$ ), then with probability close to 1 it will have no reducible fibers, and also that it will not have additive reduction anywhere.

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## Sections

As before, write an arbitrary section as

$$(x, y) = (gt^2 + at + b, ht^3 + ct^2 + dt + e).$$

The specialization to the multiplicative fiber  $\mathbb{G}_m$  at  $\infty$  is given by  $u = (h + g)/(h - g)$ .

Specialization is a multiplicative homomorphism:  $u(P + Q) = u(P)u(Q)$ .

As before, one attempts to eliminate all the other variables  $a, \dots, h$  in favor of  $u$ . Unfortunately, the resulting single polynomial equation (with coefficients involving the eight variables  $p_0, \dots, q_4$ ) is **too large** to fit in memory, and would take **too long** to compute anyway. So this system of equations is tremendously more complicated to solve, compared to the additive case.

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# Multiplicative universal equation

However, if we specialize  $p_0, \dots, q_4$  to say integers, the elimination proceeds smoothly. So, we can use **Lagrange interpolation** to (provably) read out the coefficients of the resulting polynomial equation for  $u$ , one by one (at least, the top few coefficients).

After such a massive calculation, we find a monic polynomial equation  $\Phi_\lambda(u)$  of degree 240 with coefficients in  $k[p_0, \dots, q_4]$ , which is one of the key steps in proving our theorem.

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## Multiplicative excellent family

As before,  $\mathcal{X} \rightarrow \mathbb{A}^n$  is a family of algebraic varieties, varying with respect to  $\lambda = (\lambda_1, \dots, \lambda_n)$ , and  $\mathcal{C}(X_\lambda)$  is a group of algebraic cycles on  $X_\lambda \times_{k_0} k$ , stable under  $\text{Gal}(k/k_0)$  and isomorphic to a fixed abelian group  $M$ . The fields  $k_0$  and  $k$  are as before, and we have a Galois representation

$$\rho_\lambda : \text{Gal}(k/k_0) \rightarrow \text{Aut}(\mathcal{C}(X_\lambda)) \rightarrow \text{Aut}(M).$$

The action of  $G$  on  $M$  gives rise to a “multiplicative” or “monomial” action of  $G$  on the group algebra  $\mathbb{Q}[M]$ , and we will be interested in the polynomials on this space which are invariant under  $G$ . This is the subject of [multiplicative invariant theory](#).

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# Multiplicative excellent family II

Suppose that  $G$  is a group acting on  $M$ .

## Definition

We say  $\{X_\lambda\}$  is a **multiplicative excellent family** with Galois group  $G$  if the following conditions hold:

- 1 The image of  $\rho_\lambda$  is equal to  $G$ .
- 2 There is a  $\text{Gal}(k/k_0)$ -equivariant evaluation map  $s : \mathcal{C}(X_\lambda) \rightarrow k^*$ .
- 3 There exists a basis  $\{Z_1, \dots, Z_n\}$  of  $\mathcal{C}(X_\lambda)$  such that if we set  $u_i = s(Z_i)$ , then  $u_1, \dots, u_n$  are algebraically independent over  $\mathbb{Q}$ .
- 4  $\mathbb{Q}[u_1, \dots, u_n, u_1^{-1}, \dots, u_n^{-1}]^G = \mathbb{Q}[\lambda_1, \dots, \lambda_n]$ .

# Multiplicative invariant theory of Weyl groups

The Weyl group now acts on the specializations  $u_1, \dots, u_{240}$ , but multiplicatively.

In classical notation, we can think of the (additive) lattice  $E_8$  spanned by minimal vectors  $v_1, \dots, v_8$ , and let  $u_i = e^{v_i}$ . Let the Weyl group act by  $g \cdot e^{v_i} = e^{g \cdot v_i}$  where the action in the exponent is the usual (linear action). Algebraically, we would write  $u_i/u_j$  instead of  $e^{v_i - v_j}$ , etc. If  $v_1, \dots, v_8$  is a basis of  $E_8$ , then all the  $u_i$  are Laurent polys in  $u_1, \dots, u_8$ .

It's a classical theorem (Bourbaki) that the multiplicative invariants of the Weyl group with respect to the weight lattice are a free algebra over the Weyl group orbit sums of a set of fundamental weights, or alternatively, over the fundamental characters of the Lie group (here,  $E_8$ ).

For example, the fundamental character corresponding to the 248-dimensional representation is just  $8 + \sum_{i=1}^{240} u_i$ . (The others are a lot more complicated to write down).

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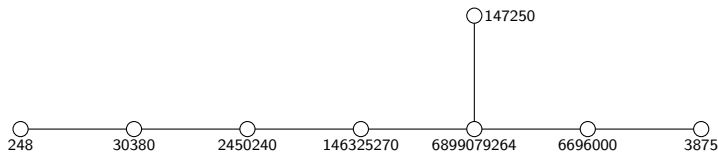
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# Fundamental weights

The **fundamental characters** are the characters of irreducible representations attached to the fundamental weights, i.e. the elements of the weight lattice which span its intersection with the Weyl chamber. In the case of  $E_8$  the fundamental representations have the following dimensions

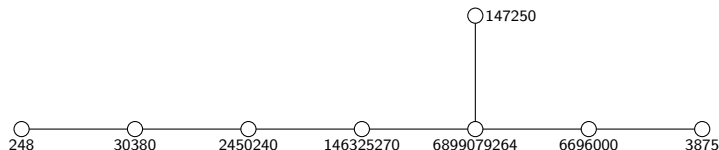


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# Transformations

This expectation turns out to be true, but it's not so easy to prove! For instance, we don't seem to find a set of 8 equations relating the Weierstrass coefficients and the (multiplicative) elementary symmetric polynomials  $\sigma_i$  (i.e. the coefficients of  $\Phi_\lambda(u)$ ) in an invertible way. Here are some of these equations:

$$\sigma_1 = 16(4q_4 - 15)$$

$$\sigma_2 = -8(1904q_4 - 512q_3 - 576p_2 - 3585)$$

$$\sigma_3 = -16(7168q_4^2 - 12288p_2q_4 - 112812q_4 + 43264q_3 - 16384q_2 + 4096p_2^2 + 64128p_2 - 14336p_1 + 142205)$$

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## Proof idea

Nevertheless, we can prove the result, using the elementary symmetric polynomials as an intermediary, whose function field is indeed equal to  $k(p_0, \dots, q_4)$ . We already have relations between these and the Weierstrass coefficients.

We then established a set of relations between the  $\sigma_i$  and the fundamental weights  $\chi_i$  by two different methods: one conceptual, using the decomposition of alternating powers of the 248-degree representation (Plethysm problem), and one directly, using a massive calculation with Laurent polynomials (and interpolation!)

Finally, compare these sets of equations and solve.

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# Genericity

To carry out these eliminations, we need a genericity condition on the Weierstrass coefficients, which ensures that the Mordell-Weil lattice is indeed  $E_8$ , i.e. that there are no reducible fibers.

The condition turns out to be  $\Phi_\lambda(1) \neq 0$ . Note that this just says that no roots of the  $E_8$  lattice specialize to zero (or multiplicatively, 1), since

$$\Phi_\lambda(1) = \prod (e^\alpha - 1)$$

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## An explicit example

Let  $e = (2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19) = 9699690$ . Consider the rational elliptic surface with coefficients

$$q_4 = -1033703267820969101/(2^5 e^2)$$

$$q_3 = 142357014158533700694389884663/(2^{10} 3e^3)$$

$$q_2 = -34310279522244459717186082618259731881473588831/(2^{22} 3^6 5^4 e^3)$$

$$q_1 = 5676600588537965616702966403319321321308511341667701463/(2^{30} 3^3 5^3 e^5)$$

$$q_0 = 212844466046687621401152105790083197307871666672878805150140946045394203/(2^{42} 3^9 5^6 e^7)$$

$$p_2 = 12325471747041766042007/(2^6 3^3 5^2 e^2)$$

$$p_1 = -734950761836067484129800296795877/(2^{20} 5e^3)$$

$$p_0 = 24098083883224058864567860980219315875451095137/(2^{23} 3^5 5^4 e^4)$$

The Mordell-Weil lattice is of rank 8 with generators for which  $u_i$  are  $\{3, 5, 7, 11, 13, 17, 19, 15/2\}$ .

The  $x$ -coordinates of the sections are

$$x(P_1) = 3t^2 - (99950606190359/620780160)t \\ + 4325327557647488120209649813/2642523476911718400$$

$$x(P_2) = (5/4)t^2 - (153332163637781/1655413760)t \\ + 5414114237697608646836821/5138596941004800$$

$$x(P_3) = (7/9)t^2 - (203120672689603/2793510720)t \\ + 6943164348569130636788638639/7927570430735155200$$

$$x(P_4) = (11/25)t^2 - (8564057914757/147804800)t \\ + 115126372233675800396600989/155442557465395200$$

$$x(P_5) = (13/36)t^2 - (347479008951469/6385167360)t \\ + 157133607680949617374030405417/221971972060584345600$$

$$x(P_6) = (17/64)t^2 - (1327421017414859/26486620160)t \\ + 5942419292933021418457517303/8901131711702630400$$

$$x(P_7) = (19/81)t^2 - (489830985359431/10056638592)t \\ + 46685137201743696441477454951/71348133876616396800$$

$$x(P_8) = (120/169)t^2 - (30706596009257/440806080)t \\ + 76164443074828743662165466409/55823308449760051200.$$

## An $E_7$ example with big Galois

Consider the rational elliptic surface with Weierstrass equation

$$y^2 + txy = x^3 + (1 + t + t^2)x + (1 + t + t^2 + t^3 - t^4).$$

It has an  $I_2$  fiber at  $t = \infty$  and Mordell-Weil lattice  $E_7$ . The splitting field is generated by a root of the following monic polynomial of degree 56, and its Galois group is  $W(E_7)$ .

# An $E_7$ example with big Galois II

$$\begin{aligned}f(X) = & X^{56} - X^{55} + 40X^{54} - 22X^{53} + 797X^{52} - 190X^{51} + 9878X^{50} - 1513X^{49} \\ & + 82195X^{48} - 17689X^{47} + 496844X^{46} - 175584X^{45} + 2336237X^{44} \\ & - 1196652X^{43} + 8957717X^{42} - 5726683X^{41} + 28574146X^{40} \\ & - 20119954X^{39} + 75465618X^{38} - 53541106X^{37} + 163074206X^{36} \\ & - 110505921X^{35} + 287854250X^{34} - 181247607X^{33} + 420186200X^{32} \\ & - 243591901X^{31} + 518626022X^{30} - 278343633X^{29} + 554315411X^{28} \\ & - 278343633X^{27} + 518626022X^{26} - 243591901X^{25} + 420186200X^{24} \\ & - 181247607X^{23} + 287854250X^{22} - 110505921X^{21} + 163074206X^{20} \\ & - 53541106X^{19} + 75465618X^{18} - 20119954X^{17} + 28574146X^{16} \\ & - 5726683X^{15} + 8957717X^{14} - 1196652X^{13} + 2336237X^{12} \\ & - 175584X^{11} + 496844X^{10} - 17689X^9 + 82195X^8 - 1513X^7 \\ & + 9878X^6 - 190X^5 + 797X^4 - 22X^3 + 40X^2 - X + 1,\end{aligned}$$

## Final comments and related work

We can similarly use the theory to produce a  $W(E_8)$  extension of  $\mathbb{Q}$  where the Weyl group acts multiplicatively on the set of roots of a degree 240 polynomial. This was carried out by Jouve, Kowalski and Zywna directly (without any reference to elliptic surfaces).

One can also use our results to produce families with multiplicative reduction and essentially every possible MW lattice permitted by Oguiso-Shioda's table, using embeddings of lattices/Dynkin diagrams.

Rather amazingly, in 2003 physicists Eguchi and Sakai wrote down an elliptic surface with the same form as ours and which can be Weierstrass-transformed to this family, with the correct coefficients in terms of the fundamental characters. They use some arguments from mirror symmetry ("Seiberg-Witten curve") to write down an ansatz, and use specializations of the family to arrive at the form.



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We can similarly use the theory to produce a  $W(E_8)$  extension of  $\mathbb{Q}$  where the Weyl group acts multiplicatively on the set of roots of a degree 240 polynomial. This was carried out by Jouve, Kowalski and Zywna directly (without any reference to elliptic surfaces).

One can also use our results to produce families with multiplicative reduction and essentially every possible MW lattice permitted by Oguiso-Shioda's table, using embeddings of lattices/Dynkin diagrams.

Rather amazingly, in 2003 physicists Eguchi and Sakai wrote down an elliptic surface with the same form as ours and which can be Weierstrass-transformed to this family, with the correct coefficients in terms of the fundamental characters. They use some arguments from mirror symmetry ("Seiberg-Witten curve") to write down an ansatz, and use specializations of the family to arrive at the form.

**Reference:** “Multiplicative excellent families of elliptic surfaces of type  $E_7$  or  $E_8$ ”, Abhinav Kumar and Tetsuji Shioda, arXiv:1204.1531 (to appear in Algebra and Number Theory).

Thank you!